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By

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and
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We wish to express our appreciation to Professor J. S. Kinney for his advice and guidance in the compilation of this thesis.

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INTRODUCTION

OBJECT

This work is an extension of the investigation by Cain[1] into the practicability of applying the principles of moment distribution to the solution of arched bents. Cain developed curves for fixed end moment, carry-over-factor, and thrust in arches of 25' span for various rise/span ratios.

We have determined that a direct linear relationship exists between these factors for any span length and have provided curves for a parabolic arch based on a 1' span from which the values for any span can be readily computed.

[1]CAIN, M. G., Application of Moment Distribution to Arched Bents, Thesis, RPI February, 1947

ABSTRACT

A method of applying the principles of mement distribution to parabolic arched bents is presented here-in.

The primary object has been to plot influence lines for the fixed-end moments, the horizontal reactions, and the vertical reactions at the ends of fixed-arches of any span length and rise/span ratio. These curves were computed by the conjugate structure method of analysis developed by Joseph S. Kinney, D.C.E.; Associate Professor of Structural Engineering, Rensselaer Polytechnic Institute, Troy, N.Y.

The authors claim no originality in regard to the concept of applying moment distribution to arches since it was recognized by Hardy Cross when he first presented his method, and is the basis of a similar work presented by Cain for one span length only.

DEFINITIONS

Carry-Over Factor.....The relationship betweenthe moment at the fixed end of a member and the moment applied at the opposite end which is hinged. Numerically, it is equal to the induced moment at the fixed end when a unit moment is applied at the hinged end.

Absolute Stiffness... The moment required to produce unit rotation of the hinged end of a member, whose far end is fixed.

Fixed End Moment..... The moment which would exist at the ends of a member if its ends were rigidly fixed against rotation.

Distribution Factor.. The ratio of the stiffness of one of the members at a joint to the sum of the stiffnesses of all the members at the joint.

Δ_H..... Total horizontal deflection of a point

Δ_V..... Total vertical deflection of a point

θ..... Total angular rotation of a point

δ_{VH}..... Vertical deflection of a point due to the application of a unit horizontal force at the point.

δ_{VV}..... Vertical deflection of a point due to the application of a unit vertical force.

δ_{VM}..... Vertical deflection of a point due to the application of a unit moment.

δ_{HH}..... Horizontal deflection of a point due to the application of a unit horizontal force.

δ_{HV}..... Horizontal deflection of a point due to the application of a unit vertical force.

δ_{HM}..... Horizontal deflection of a point due to the application of a unit moment.

$\delta_{\theta H}$ ..... Angular rotation of the member at a point due to the application of a unit horizontal force at the point.

$\delta_{\theta V}$ ..... Angular rotation of the member at a point due to the application of a unit vertical force at the point.

$\delta_{\theta M}$ ..... Angular rotation of the member at a point due to the application of a unit moment at the point.

S ..... Effective span of the arch in feet.

$\widehat{L}$ ...... Total arc length of the arch in feet.

ds ..... Arc length of a segment of the arch in feet.

y ..... Distance from the crown of the arch to the neutral point in feet.

x ..... Horizontal distance from a reference point, usually the left end of the arch, to the center of the arch segment in feet.

x' ..... The horizontal distance from the neutral point to the point on the arch where the M/EI loading on an arch segment is assumed to be concentrated, measured in feet.

y ..... The vertical distance from a reference point, usually the left springing, to the center of the arch segment in feet.

y' ..... The vertical distance from the neutral point to the point on the arch where the M/EI loading on an arch segment is assumed to be concentrated, measured in feet.

Rise ... The vertical distance from the springing to crown in feet.

SIGN CONVENTION

The carry over factor is considered positive when the applied moment at the hinged end produces tension in the top of the member and the induced moment at the fixed end produces tension in the bottom of the member.

In the tabulated results for the fixed end moments, a positive moment is that moment which causes tension in the top fibers of the member.

In using the moment distribution method of solution, a positive moment is that internal moment which tends to rotate the joint in a clockwise direction.

PROCEDURE

As pointed out by Hardy Cross in the original presentation of his method of solving indeterminate rigid frames by "moment distribution", "It makes no difference whether the members are of constant or of varying cross-section, curved or straight, provided the constants: (a) stiffness at each end, (b) fixed-end moments at each end, and (c) carry-over factor at each end, are known or can be determined."

Since many structures have been constructed with curved top members on rigid frames, and since the usual method of solution for the stresses in such members is long and tedious, it appears that any information that will permit the engineer to apply the simple method of moment distribution to their solution will be of considerable value. We feel that there is much work to be done along this line and we have simply gone one step further than Cain in providing data which permits application of the method to spans of any length and for many rise/span ratios. Future investigation to determine these constants for any practical rise/span ratio and for members of varying cross-section is desirable.

In our work we have dealt with a parabolic arch of constant moment of inertia, and rise/span values of 0.04, 0.08, 0.20, 0.30, 0.40. Solutions were obtained for 20 and 30 foot spans, and it was determined that all the required constants varied in some direct relationship with the span of the arch. Consequently, all tabulated results and curves are based upon a unit span of 1'-0".

The values obtained were found by the use of "The Conjugate Structure Method " of analysis developed by Joseph Sterling Kinney, D. C. E.; Associate Professor of Structural Engineering, Rensselaer Polytechnic Institute, Troy, N. Y. Its method of application is described below.

"The conjugate structure, for a given real structure, is identical, in the lengths of its members and their relative position, to the real structure. It is considered to be positioned, however, so that it is located in a horizontal plane with the loads of the M/EI diagrams acting in a vertical direction.

The end of the conjugate structure, which corresponds to the end of the real structure which is permitted to deflect, if such an end exists, is always considered to be fixed in the conjugate structure.

The moment at any point on the conjugate structure (such moment being that caused by the M/EI diagram

loading) is the deflection of the corresponding point on the real structure in a direction perpendicular to the lever arms causing the moment in the conjugate structure.

The shear at any section of the conjugate structure is the rotation of the corresponding section of the real structure.

Sign Convention (for the conjugate structure)

Any moment in the real frame which causes compression of the inside surface thereof is considered to result in an M/EI diagram acting down on the conjugate structure.

If the moment at any section of the conjugate structure results in tension on the top fiber, the vertical deflection at the corresponding section of the real structure is down and the horizontal deflection is such as to shorten the horizontal projection of the distance between given points on the real structure.

If a section be passed through any point on the conjugate structure and if the portion of the conjugate structure thereby cut free tends to move down, then the angular rotation of the corresponding point on the real structure is counterclockwise."[1]

1. Course Notes of Professor J. S. Kinney, 9/25/47

FIXED END MOMENT AND THRUST DETERMINATION:

In the determination of the fixed end moments, we decided to use a combination of the Neutral Point Method and the Conjugate Structure Method, thereby eliminating the need of solving simultaneous equations.

The arch is cut at the left reaction, cantilevering out from the right end, and a weightless bracket of infinite cross-section rigidly connects the cut end with the neutral point.

The arch is divided into ten equal horizontal projections, and the length of each arc segment determined as shown in appendix "A". It will be noted that two methods were used in the calculation of these arc lengths. Method "A" was used for the flat arches up to and including 0.20, and consists of using the first few terms of the infinite series which expresses the arc length of a parabola. The expressions used and a sample of the calculations appear in appendix "A". Method B was used for the higher values of rise/span since it was found that too many terms of the infinite series were required to accurately determine the arc lengths. Method "B" uses the exact logarithmic expression for arc length and sample calculations are to be found in Appendix "B".

The neutral point, the centroid of the conjugate structure, was computed by taking moments of the elastic areas about an axis through the crown as indicated in appendix "C". In the tables for computation of the location of the centroids it will be noted that the values of Ad^2 were determined and recorded as I_{yy} and I_{xx}. In normal calculations a designer would neglect the effects of the I_{cg} and the rotation of the axis for each segment which would be required for an accurate computation of the Moment of Inertia values. In checking our conjugate structure solution by the analagous column method, we found it was necessary to include the effect of the I_{cg} and the rotation of the axes in determining the moments of inertia in order to make the two methods agree. For slide rule accuracy it is sufficiently accurate to use the Ad^2 value as the moment of Inertia, but in the calculator accuracy as contained herein, the error was very appreciable.

The next step in our calculations was to determine the deflections of the neutral point due to the application at the neutral point of unit horizontal and vertical loads, and a unit moment. These were recorded as δ_{VV}, δ_{HH}, & δ_{M} in which

δ_{VV} was the moment about the $\chi'-\chi'$ axis when the conjugate structure was loaded with the M/EI diagram for a unit vertical load at the neutral point; δ_{HH} was the moment about the $\chi'-\chi'$ axis, when the conjugate structure was loaded with the M/EI diagram for a unit horizontal load at the neutral point; and $\delta_{\theta M}$ was the shear at the neutral point when the conjugate structure was loaded with the M/EI diagram for a unit moment applied at the neutral point. Since the value of M/EI is a constant across the arch when a unit moment is applied at the neutral point, $\delta_{\theta M}$ was equal to the total arc length.

By definition of the neutral point, a horizontal force at the neutral point causes only horizontal deflection of the neutral point, a unit moment causes only rotation, and a unit vertical force causes only vertical deflection. All deflections being in the same direction as the applied unit force or moment. See Figure 1.

Next, unit vertical loads were placed at each of the 1/10 points across the span, and the deflections of the neutral point determined by means of the conjugate structure. These deflections were recorded as Δ_V, Δ_H, & θ

It should be pointed out at this time that in the solution of our problems, we divided the M/EI diagram loading on a given arc segment into a triangle and a rectangle, and considered the total weight of the rectangle to be concentrated at the middle of the horizontal projection and located on the arc itself. The total weight of the triangle was considered to be concentrated at the 1/3 point of the horizontal projection from the base of the triangle, and located on the arc itself at that horizontal projection. See figure 2.

From the theories of the neutral point method, the following relationships hold true:

$$H_o = -\frac{\Delta_H}{\delta_{HH}} = H_L$$

$$V_o = -\frac{\Delta_V}{\delta_{VV}} = V_L$$

$$M_o = -\frac{\theta}{\delta_{\theta M}}$$

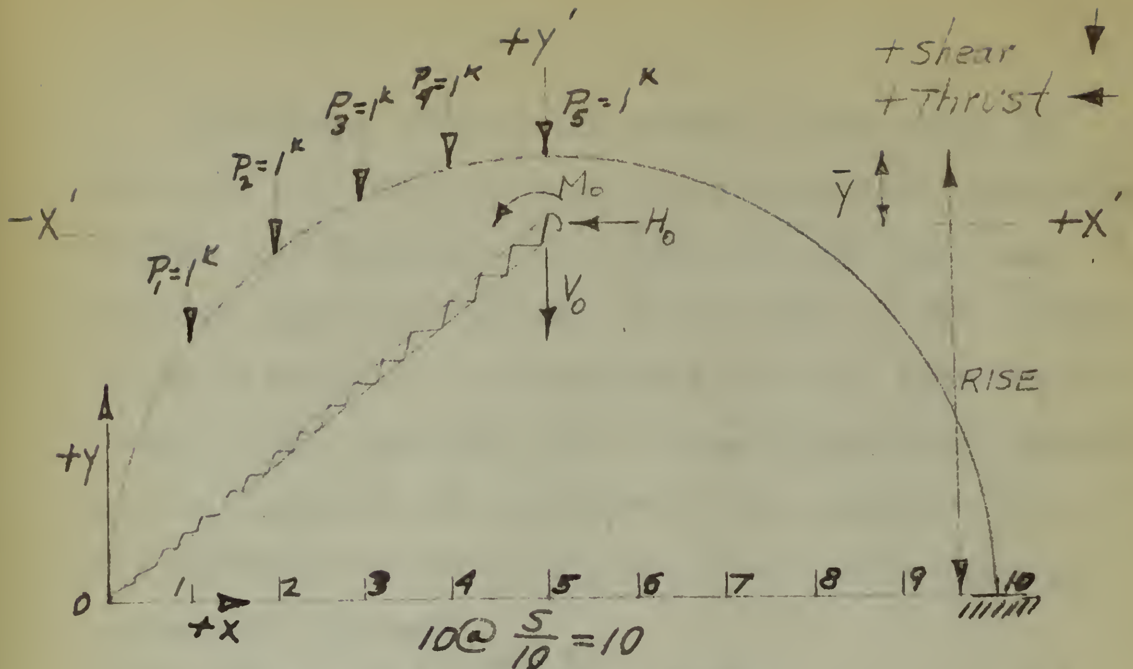


FIGURE 1 Parabolic Arch
Redundant reactions applied at neutral point and vertical loads applied at 1/10 th span points.

M/EI diagram loading on an arc--divided into rectangle and triangle sections.

moment on real structure @ A
M_2-moment on real structure @ B

Horizontal Projection of M/EI diagram loading on the arc.

W_1-elastic wt. of rect.

W_2-elastic wt. of triangle

ds-actual arc length

FIGURE 2

The Shear, thrust, and moment at any point on the real structure can readily be determined by drawing a free body diagram of the arch with the right end rigidly fixed and the left end attached to the neutral point by means of the weightless bracket, applying the loads on the structure and the reactions at the neutral point. Applying the reactions at the neutral point in the directions shown in Figure 1, and using the sign convention thereof,

$$M_L = M_o + H_o\,(RISE - \bar{y}) - V_o\,\frac{S}{2}$$

$$M_R = M_o + H_o\,(RISE - \bar{y}) + V_o\,\frac{S}{2} + P\,(S - x)$$

with a positive moment indicating compression on the inside of the arch.

Sample calculations for the determination of the fixed end moments, vertical shear, and horizontal reactions for various locations of unit loads on the arch are shown in Appendix "D". Tabulated results for the various rise/span ratios, with all values expressedin terms of the span length in feet, are shown on Page 52.
The influence lines for the fixed end moments, the horizontal thrusts and the vertical shears at the springing for the various rise/span ratios are shown in Figures 3 through 10 inclusive, pages 20-24.

DETERMINATION OF CARRY-OVER FACTOR

When an arch of constant moment of inertia is cut at the left springing, and redundant forces applied as shown in the figure below, deflection of the left springing will be as follows:

$$(1)\quad \Delta_V = M_A\,\delta_{VM} + V_A\,\delta_{VV} + H_A\,\delta_{VH}$$

$$(2)\quad \Delta_H = M_A\,\delta_{HM} + V_A\,\delta_{HV} + H_A\,\delta_{HH}$$

$$(3)\quad \theta = M_A\,\delta_{\theta M} + V_A\,\delta_{\theta V} + H_A\,\delta_{\theta H}$$

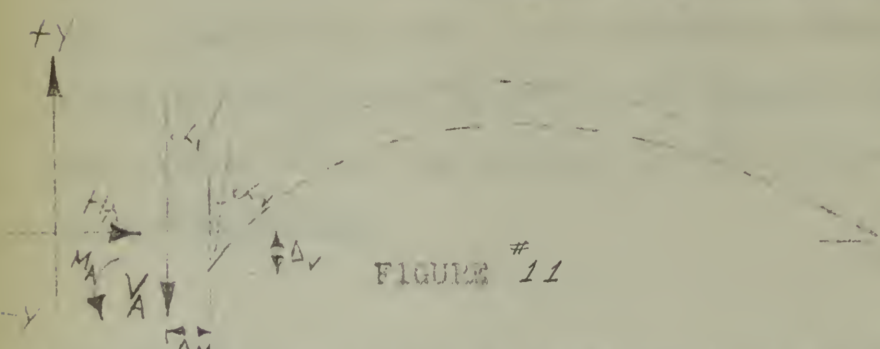


FIGURE #11

Multiplying both sides of equations 1, 2, &3 by EI results in:

$$(4)\quad EI\Delta_V = M_A\,(EI\,\delta_{VM}) + V_A\,(EI\,\delta_{VV}) + H_A\,(EI\,\delta_{VH})$$

$$(5)\quad EI\Delta_H = M_A\,(EI\,\delta_{HM}) + V_A\,(EI\,\delta_{HV}) + H_A\,(EI\,\delta_{HH})$$

$$(6)\quad EI\theta = M_A\,(EI\,\delta_{\theta M}) + V_A\,(EI\,\delta_{\theta V}) + H_A\,(EI\,\delta_{\theta H})$$

From the theories of virtual work and Maxwells Law of reciprocal deflections:

$$EI\delta_{VM} = EI\delta_{\theta V} = \Sigma X ds = \widehat{L}\cdot\frac{S}{2}$$

$$EI\delta_{HM} = EI\delta_{\theta H} = \Sigma y ds = \widehat{L}\cdot(RISE - \bar{y})$$

$$EI\delta_{\theta M} = \Sigma ds = \widehat{L}$$

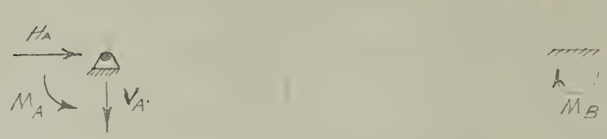
$$EI\delta_{VH} = EI\delta_{HV} = \Sigma Xy ds = \widehat{L}\cdot\frac{S}{2}(RISE - \bar{y})$$

$$EI\delta_{VV} = \Sigma X^2 ds$$

$$EI\delta_{HH} = \Sigma y^2 ds$$

Since the carry-over factor is numerically equal to the moment induced at the fixed end of a member when a unit moment produces rotation at the hinged end, there will be no vertical or horizontal deflections. Consequently equations (4) and (5) are equal to zero.

Solving equations (4) & (5) simultaneously will give us the forces induced at the hinged end when a unit moment is applied. The carryover factor will be found by setting up the free body diagram indicated in the sketch below, and solving for the induced moment at the fixed end.



When the induced moment at the fixed end is such that it causes tension on the same side of the member at the fixed end as the applied moment causes on the hinged end, the carry-over factor is considered negative.

It is evident that the application of a moment at the hinged end will introduce a horizontal thrust. This thrust must be taken into account in the solving of an arched bent by moment distribution since every time a

joint is rotated and the moments balanced out at the joint, a moment will be applied to the arch. Such applied moment will change the horizontal restraining force at the ends of the arch.

Sample calculations for the determination of the carry-over factor and horizontal restraining forces will be found on page 3 of Appendix "E". The carry over factors and horizontal restraining forces are tabulated in Appendix "E", in terms of the length of span, and rise/span ratios.

ABSOLUTE STIFFNESS

The absolute stiffness is defined as the moment required to produce unit rotation of the hinged end of a member, whose far end is rigidly fixed. Thus, there will be no horizontal or vertical deflection at the hinged end, and equations (4) & (5) are again equal to zero.

Solving these equations simultaneously will give us the forces induced at the hinged end in terms of the applied moment. Substituting these values in equation (6) will give us $EI\theta$ expressed in terms of the applied moment. Setting $\theta = 1$ for unit rotation, and solving for M_A, we will get the absolute stiffness of the arch. After solving several span lengths with the same rise/span ratio, it was determined that the absolute stiffness varied inversely as the span. Consequently the tabulated

values of the absolute stiffness for each rise/span ratio are in terms of EI/S. Sample calculations for the determination of the absolute stiffness are shown on Page 3 of Appendix "E".

SPREAD

It has been shown in the calculations for the fixed end moments that every load placed on the arch causes a horizontal thrust at the end. This thrust will cause spread in the arched bent and must be taken into consideration when solving the arched bent by moment distribution.

If a spread of Δ_H feet occurs while there is no vertical deflection or rotation of the joint, reactions will be induced at the ends of the arch. These reactions will be determined by setting equations (4) & (6) equal to zero and solving equations (4), (5), & (6) simultaneously for H_A, V_A, and M_A in terms of EI .
After solving several spans for the same rise/span ratio, it was determined that V_A was equal to zero in all cases and that M_A varied inversely as the square of the span, and H_A varied inversely as the cube of the span.
Therefore the induced moment was tabulated in terms of $\frac{EI\Delta_H}{S^2}$ and the thrust in terms of $\frac{EI\Delta_H}{S^3}$ for all values of Rise/Span. Results are tabulated on page 57 and sample calculations are shown on page 56 .

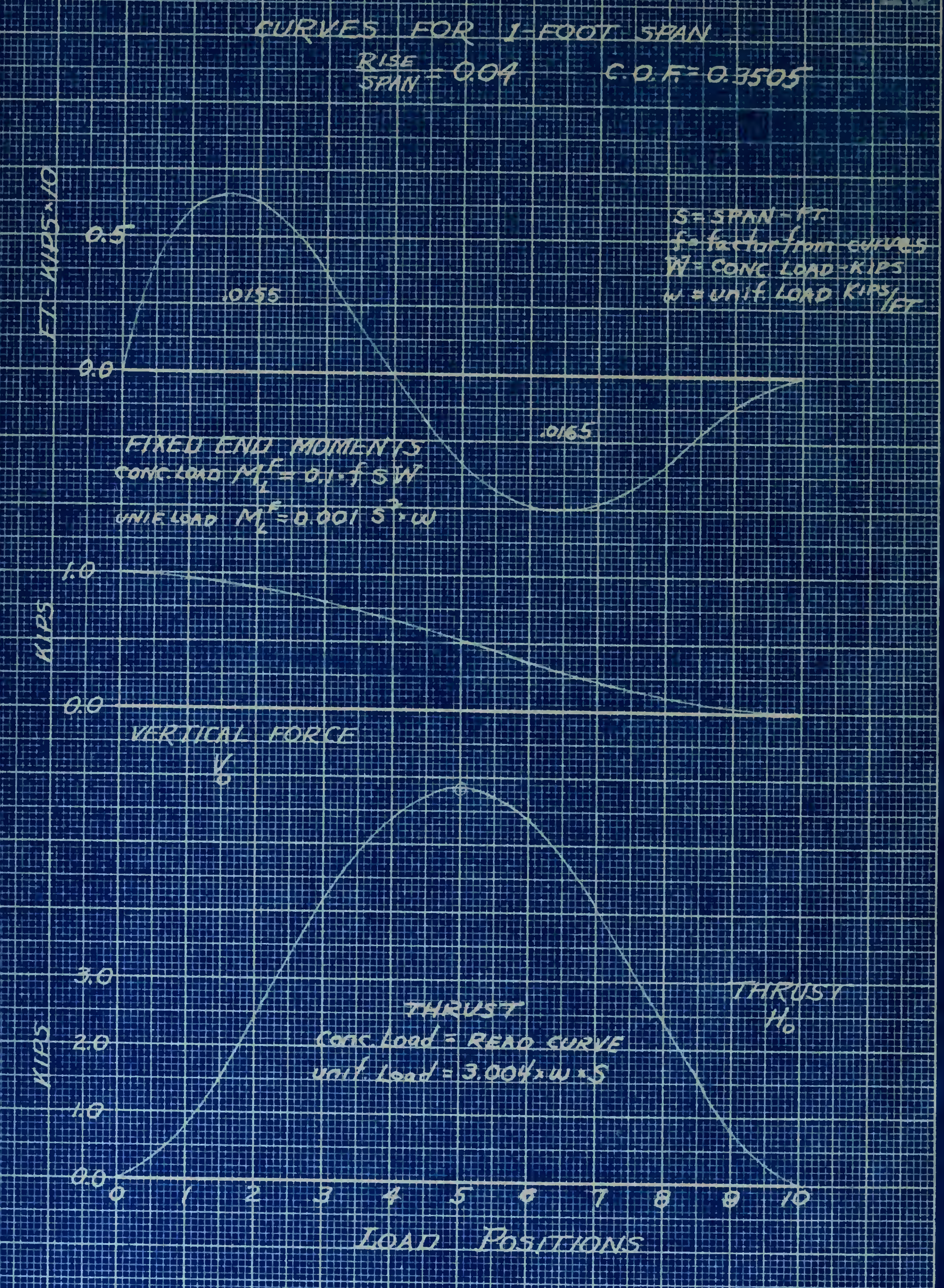
CONCLUSIONS

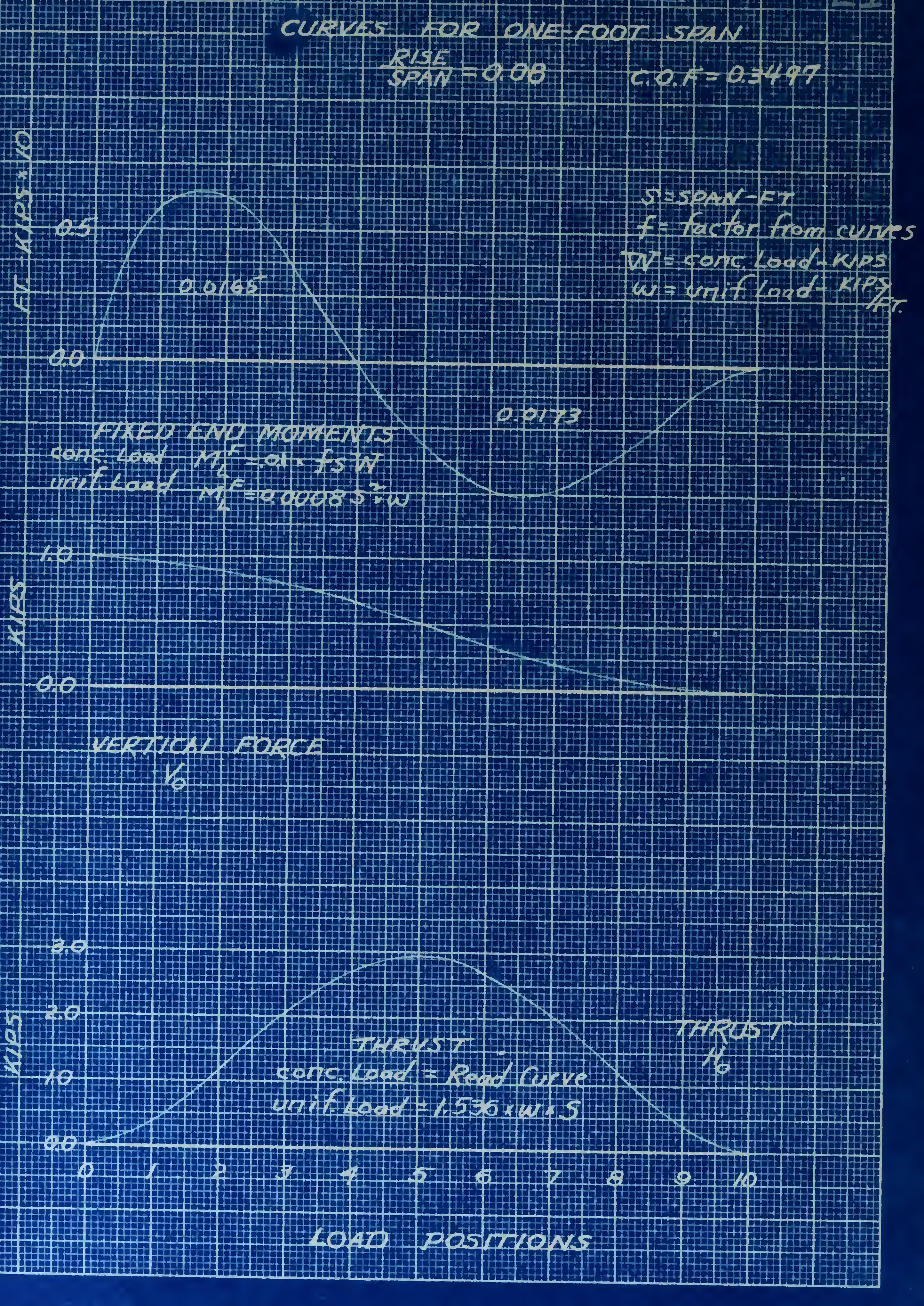
In this paper, we have presented curves and tabulated data which will permit an engineer to solve any parabolic arched bent by moment distribution regardless of loading, provided the arched member is of constant moment of inertia and has rise/span ratios between 0.04 and 0.40.

When similar curves and data are determined for arched members with varying moment of inertia across the arch, all parabolic arched bents may be solved with great saving in time and energy by the method of moment distribution.

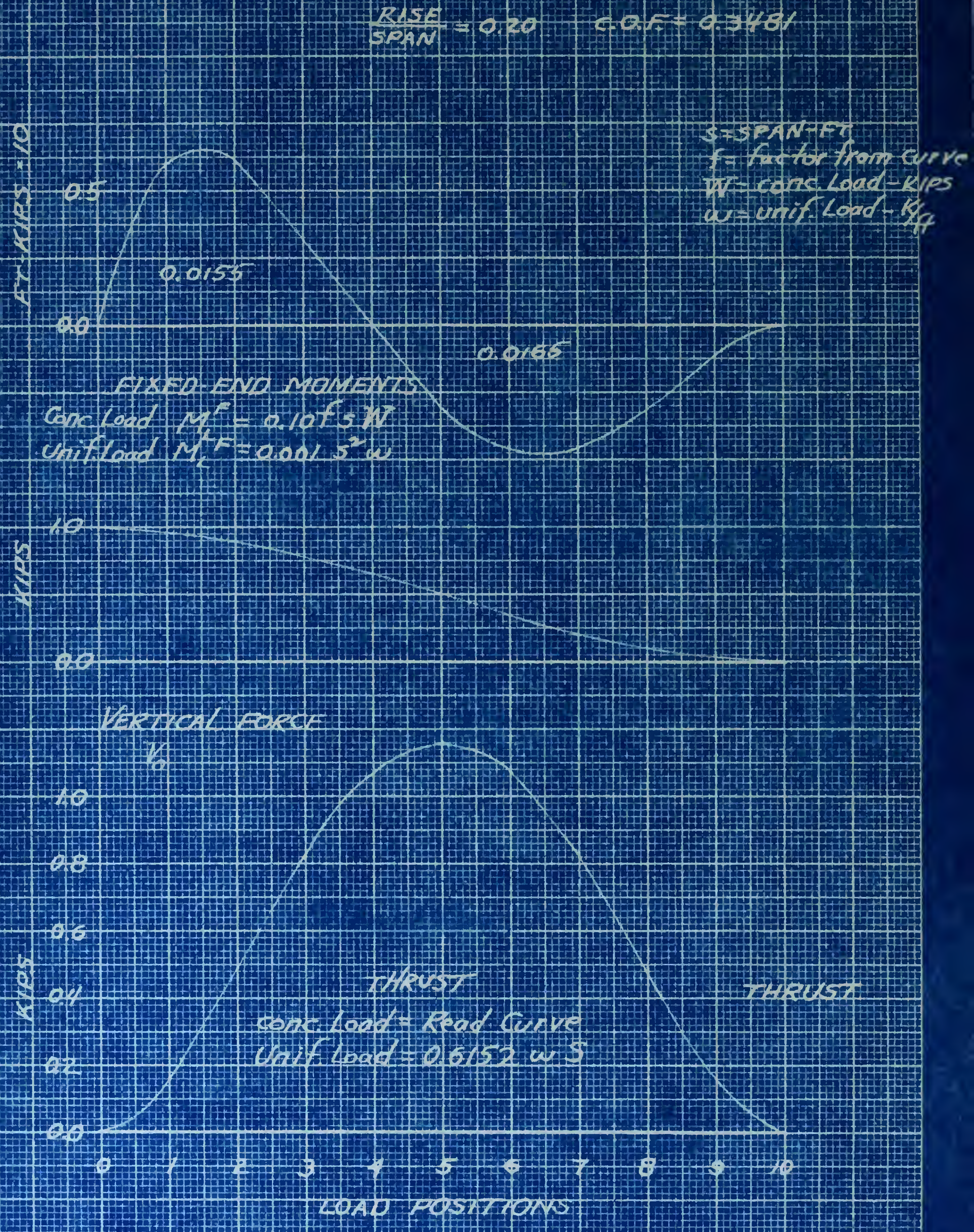
As a result of this investigation, it was found that for a given rise/span ratio; the following relationships are true:

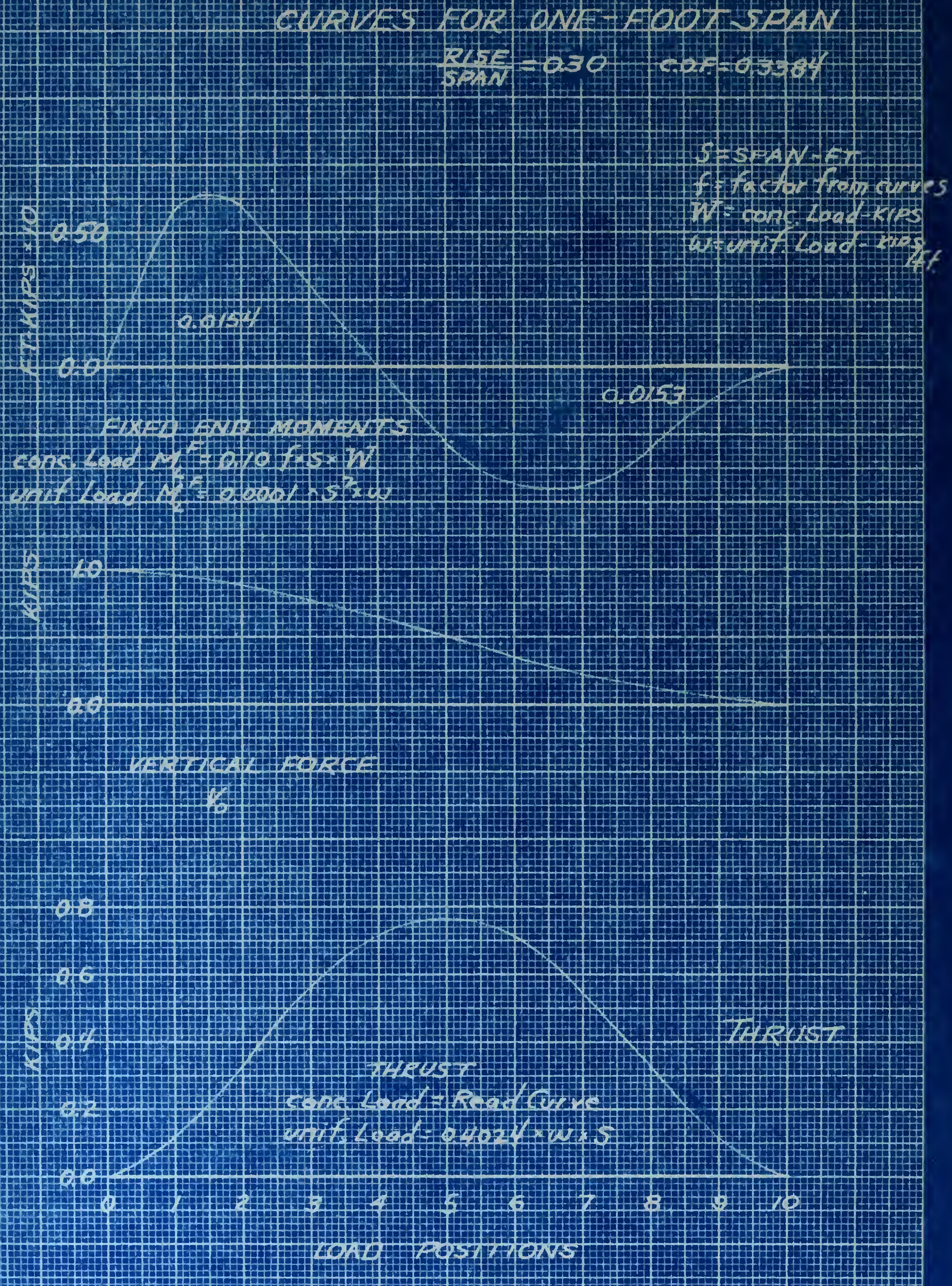
1. The shear and thrust are constant for all spans.
2. The F.E.M. is directly proportional to the span, i.e. F.E.M. for 20'span= 20 x FEM for 1' span.
3. The carry over factor is constant for all spans.
4. The absolute stiffness varies inversely as the span.

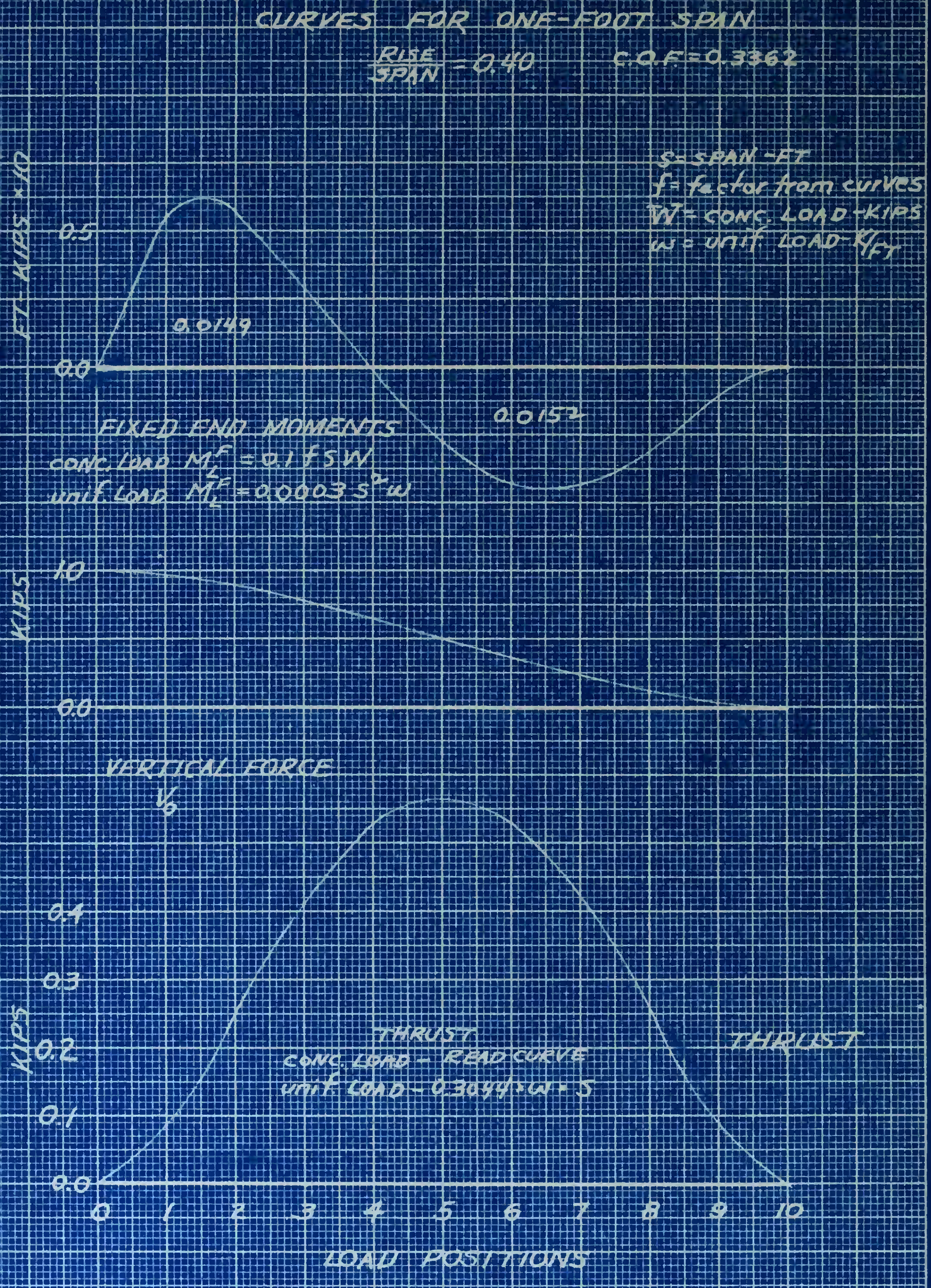

CURVES FOR 1-FOOT SPAN
RISE / SPAN = 0.04
C.O.F. = 0.3505
FT. KIPS × 10
0.5
0.0
.0155
.0165
S = SPAN - FT.
f = factor from curves
W = CONC. LOAD - KIPS
w = unit LOAD KIPS/FT
FIXED END MOMENTS
CONC. LOAD $M_L^F = 0.1 \cdot f\ S\ W$
UNIF. LOAD $M_L^F = 0.001\ S^2 \cdot w$
KIPS
1.0
0.0
VERTICAL FORCE
V_0
KIPS
3.0
2.0
1.0
0.0
THRUST
Conc. Load = READ CURVE
unif. Load = 3.004 × w × S
THRUST
H_0
0
1
2
3
4
5
6
7
8
9
10
LOAD POSITIONS


CURVES FOR ONE-FOOT SPAN
RISE/SPAN = 0.08
C.O.F = 0.3497
S = SPAN - FT
f = factor from curves
W = conc. Load - KIPS
w = unif. load - KIPS/FT.
FT. KIPS x 10
0.5
0.0165
0.0
0.0173
FIXED END MOMENTS
conc. Load $M_L^F = .01 \times fSW$
unif. Load $M_L^F = 0.0008 S^2 w$
KIPS
1.0
0.0
VERTICAL FORCE
V_o
KIPS
3.0
2.0
1.0
0.0
THRUST
conc. Load = Read Curve
unif. Load = 1.536 x w x S
THRUST
H_o
0 1 2 3 4 5 6 7 8 9 10
LOAD POSITIONS

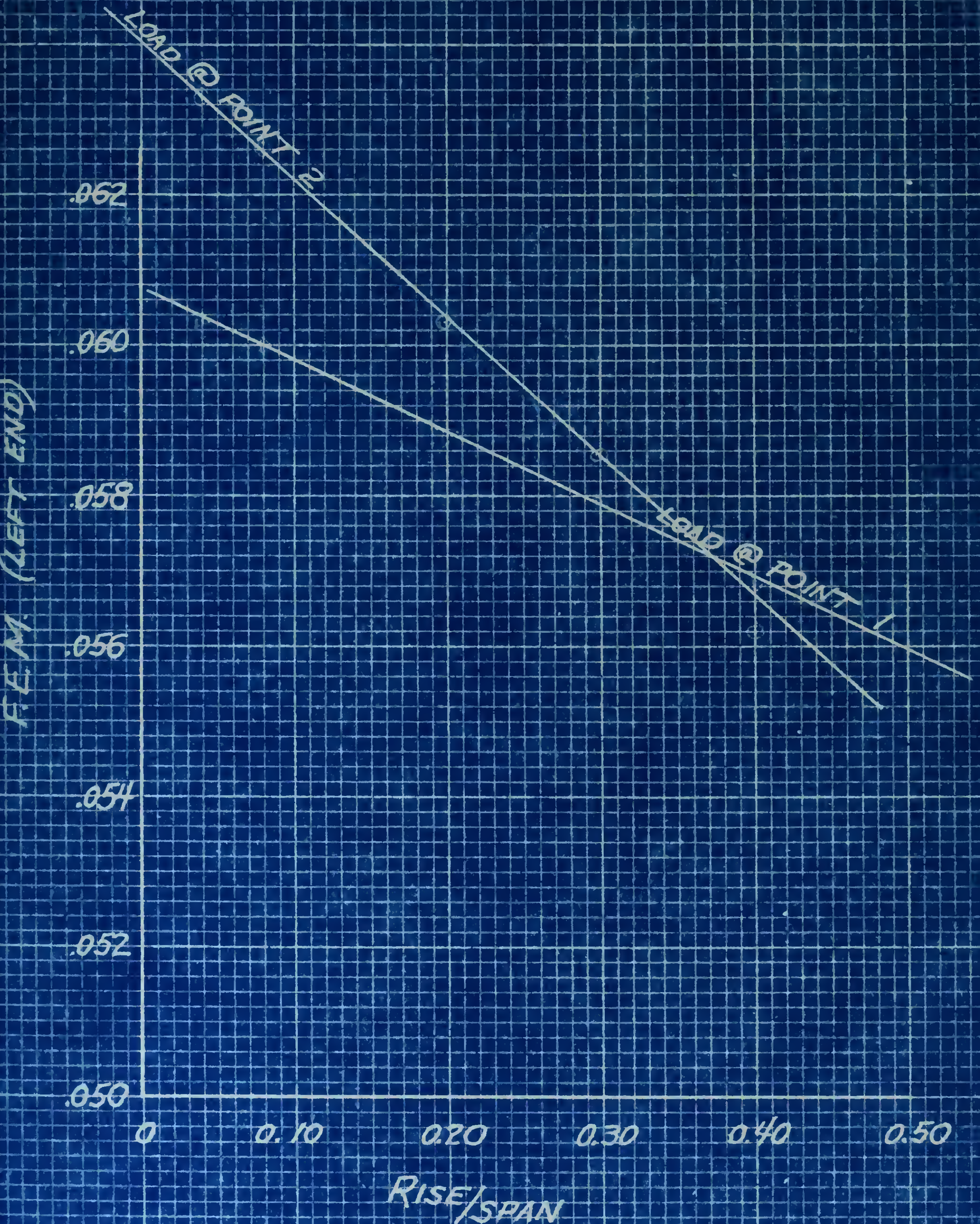
CURVES FOR ONE-FOOT SPAN




CURVES FOR ONE-FOOT SPAN
RISE/SPAN = 0.30 C.O.F. = 0.3384
S = SPAN - FT
f = factor from curves
W = conc. Load - KIPS
w = unif. Load - kips/ft
FT-KIPS × 10
0.50
0.0154
0.0
0.0153
FIXED END MOMENTS
conc. Load M^F = 0.10 f × S × W
unif. Load M^F = 0.0001 × S^2 × w
KIPS
1.0
0.0
VERTICAL FORCE
V₀
0.8
0.6
KIPS
0.4
0.2
0.0
THRUST
THRUST
conc. Load = Read Curve
unif. Load = 0.4024 × w × S
0 1 2 3 4 5 6 7 8 9 10
LOAD POSITIONS

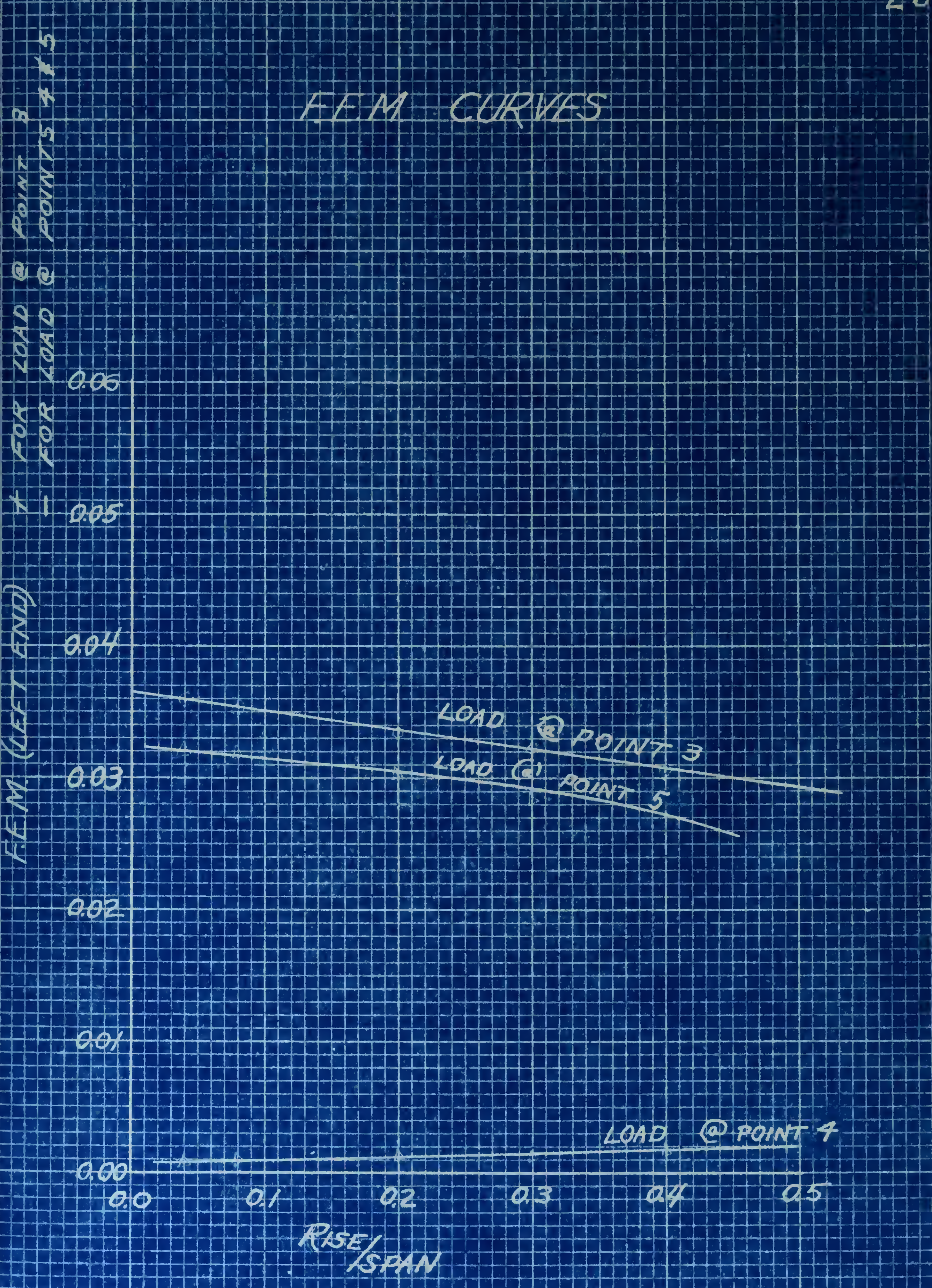

CURVES FOR ONE-FOOT SPAN
RISE/SPAN = 0.40
C.O.F. = 0.3362
S = SPAN - FT
f = factor from curves
W = CONC. LOAD - KIPS
w = unif. LOAD - K/FT
FT-KIPS × 10
0.5
0.0149
0.0
0.0152
FIXED END MOMENTS
CONC. LOAD M_L^F = 0.1 f S W
unif. LOAD M_L^F = 0.0003 S^2 w
KIPS
1.0
0.0
VERTICAL FORCE
V_0
0.4
0.3
KIPS
0.2
0.1
0.0
THRUST
CONC. LOAD - READ CURVE
unif. LOAD - 0.3044 · w · S
THRUST
0
1
2
3
4
5
6
7
8
9
10
LOAD POSITIONS

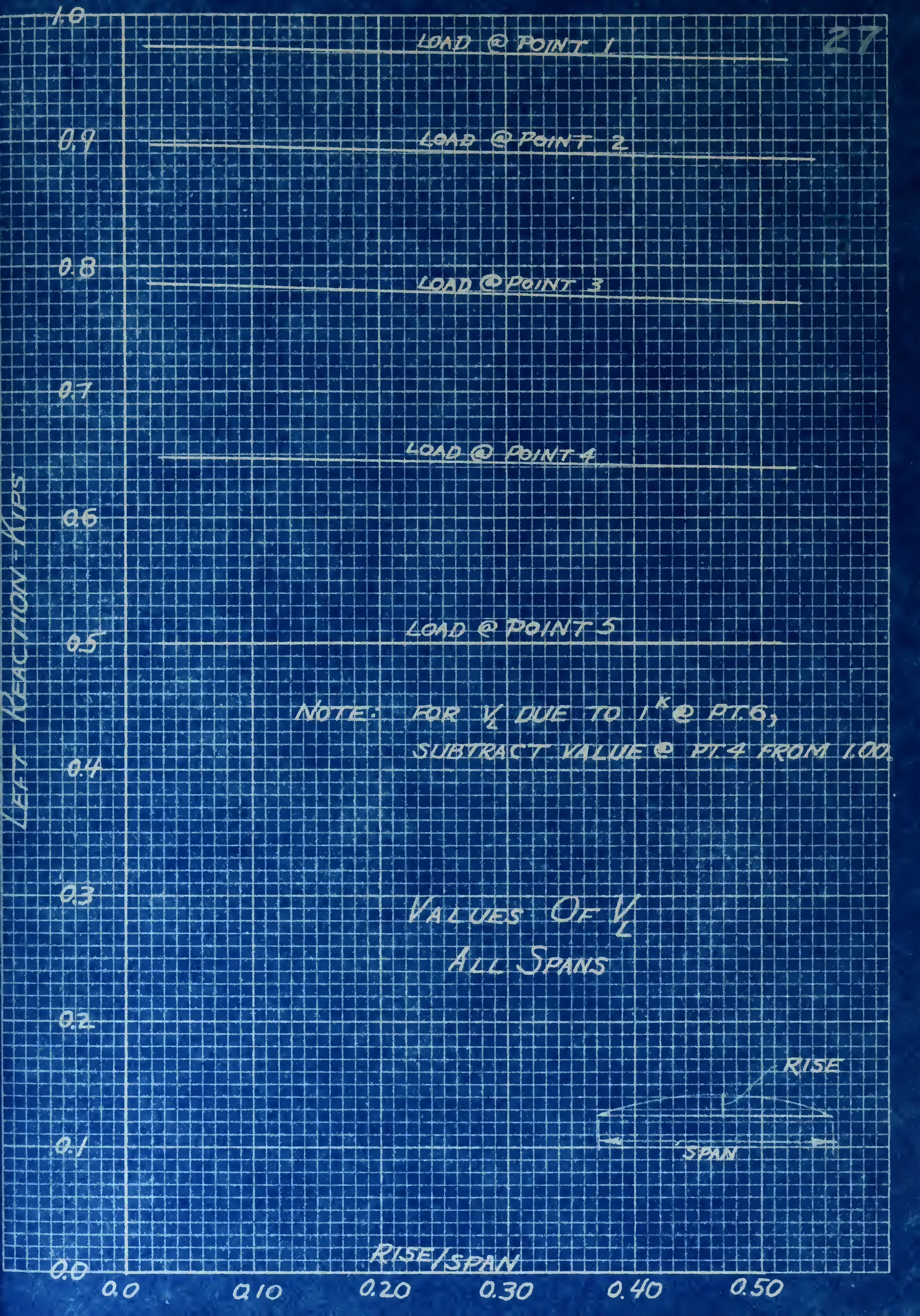
F.E.M. CURVES



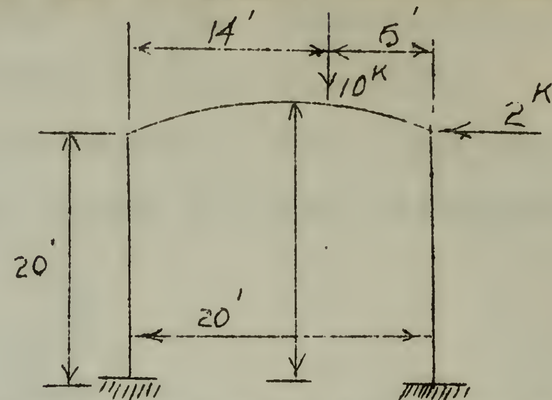
LOAD @ POINT 2
LOAD @ POINT 1
F.E.M. (LEFT END)
.062
.060
.058
.056
.054
.052
.050
0
0.10
0.20
0.30
0.40
0.50
RISE/SPAN

F.E.M. CURVES


F.E.M. (LEFT END) + FOR LOAD @ POINT 3
− FOR LOAD @ POINTS 4 & 5
0.06
0.05
0.04
0.03
0.02
0.01
0.00
LOAD @ POINT 3
LOAD @ POINT 5
LOAD @ POINT 4
0.0
0.1
0.2
0.3
0.4
0.5
RISE/SPAN


LOAD @ POINT 1
LOAD @ POINT 2
LOAD @ POINT 3
LOAD @ POINT 4
LOAD @ POINT 5
NOTE: FOR V_L DUE TO 1^K @ PT. 6,
SUBTRACT VALUE @ PT. 4 FROM 1.00
VALUES OF V_L
ALL SPANS
RISE
SPAN
LEFT REACTION - KIPS
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
RISE/SPAN
0.0
0.10
0.20
0.30
0.40
0.50

APPLICATION OF RESULTS TO THE SOLUTION OF AN ARCHED BENT BY MOMENT DISTRIBUTION.



GIVEN: An arched bent having a span of 20'-0" and a parabolic rise of 4'-0" as shown in the above sketch. All members have the same moment of Inertia. Determine the reactions at the supports and the moments at B and C, caused by the given loads on the bent.

From the curves on page 22, the following values for the fixed end arch conditions can be determined:

$$M^F_{BC} = -\frac{(0.460)\,20}{10} = -9.20'k$$

$$M^F_{CB} = +\frac{(0.334)\,20}{10} = +6.68$$

$$H_B = 8.30 \text{ kips} \longrightarrow$$

$$H_C = 8.30 \text{ kips} \longleftarrow$$

with a positive moment indicating tension on the top of the member. Since a positive moment as used in moment distribution is that moment which tends to rotate the joint clockwise,

$$M^F_{BC} = -9.20 \text{ ft.kips} \qquad M^F_{CB} = -6.68 \text{ ft. kips}$$

The absolute stiffness of the arch, from page 57, is equal to 7.8855EI/S which equals 0.3943 EI. The carry-over factor is -0.343.

The absolute stiffness of the prismatic column members is equal to $\frac{4\ EI}{L}$ which equals 0.200EI and the carry over factor is equal to + 0.500.

Assuming rotation of the joints only, solve for the moments at all joints by moment distribution.

JOINT	A		B					C			D
MEMB.	AB		BA	BC				CB	CD		DC
K	.200		.200	.3943	BALANCING MOMENTS			.3943	.200		.200
$\frac{K}{\Sigma k}$	-		.337	.663				.663	.337		-
COF		+.5				-.343			+	.50	
FEM	+1.55		+3.10	-9.20 +6.10 -1.98	+6.10		+5.80	-6.68 -2.09 +5.80	+2.97		+1.48
	+0.32		+0.64	+1.34	+1.34						
				-0.10			+0.30	-0.45 +0.30	+0.15		+0.08
			+0.03	+0.07	+0.07						
							+0.02	-0.03 +0.02	+0.01		
M_R	+1.89		+3.77	-3.77				-3.13	+3.13		+1.56
Bal. Moms.					+7.51		+6.12				

The chart shown differs from the usual moment distribution chart in that a record is kept of the balancing moments taken by the arch when the joint is permitted to rotate. This balancing moment changes the horizontal restraining force at the ends of the curved member by an amount dependent on the properties of the arch in question.

For arched bents, the required horizontal restraining forces at the ends of the arch are made up of the following components:

(a) That force necessary to restrain the horizontal thrust at the ends of the arch caused by applying a load on the arch or caused by spread in the arch.

(b) That force necessary to restrain the horizontal thrust at the ends of the arch caused by the balancing moments.

(c) That necessary to balance the shears in the column members.

(d) That force necessary to prevent sidesway when a horizontal load is applied to the bent.

A positive balancing moment (using the sign convention of moment distribution) at the left springing causes equal horizontal restraining forces at the two ends of the arch acting in a direction away from each other. A positive balancing moment at the right springing causes equal horizontal restraining forces at the ends of the arch acting in a direction toward each other.

For the bent under consideration, the horizontal restraining forces at the ends of the arch caused by the balancing moments is equal to 4.2884 M/S which equals 0.21442 M. See page 57.

The Artificial joint restraints (AJR) required to prevent the original bent from deflecting due to sidesway or spread are equal to the horizontal restraining forces at the ends of the arch.

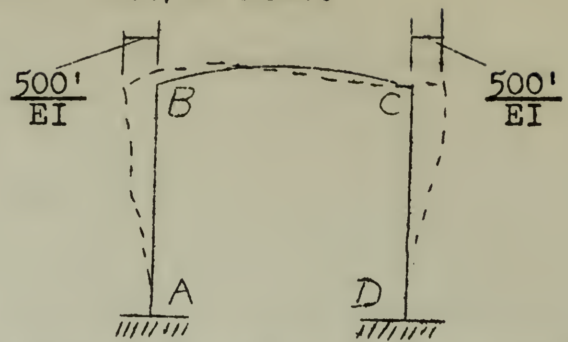
HRF caused by:	HRF_L $\xrightarrow{+}$	HRF_R
(a) Load on the arch	+8.30	-8.30
(b) Balancing Moment at B (0.21442)(7.51)	-1.61	+1.61
(c) Balancing Moment at C (0.21442)(6.12)	+1.31	-1.31
(d) Shear in member AB $\frac{(1.89 + 3.77)}{20}$	+0.28	
(e) Shear in member CD $\frac{(1.58 + 3.13)}{20}$		+0.24
(f) 2 Kip load at C		+2.00
	AJR_L +8.28 →	AJR_R -5.76 ←

If we let x equal artificial joint restraint resistant against spread and let y equal artificial joint restraint resisting sidesway, we have:

$$x + y = 8.28$$
$$x - y = 5.76$$
$$2x = 14.04$$
$$x = 7.02$$
$$y = 1.26$$

Thus the AJR_{sp} required to resist spread equals 7.02 kips, and the AJR_{ss} required to resist sidesway equals 1.26 kips.

Assume a spread of 1000/EI feet.



For a spread of the arched bent as shown, the following reactions are induced at the ends of the arch:

From Page

$$H_B = 247.6 \frac{EI\,\Delta_H}{S^3} = \frac{247.6\ EI}{8000}\frac{(1000)}{EI} = 30.95 \longleftarrow$$

$$H_C = 30.95 \longrightarrow$$

$$M^F_{BC} = +32.30 \frac{EI\,\Delta_H}{S^2} = \frac{32.3}{400} EI \frac{1000}{EI} = +80.75$$

$$M^F_{CB} = + 80.75$$

with a positive moment indicating tension in the top of the arch. Hence for moment distribution sign convention:

$$M^F_{BC} = + 80.75 \text{ ft kips} \qquad M^F_{CB} = -80.75 \text{ ft. kips.}$$

The moments induced in the ends of the column members are equal to $6\ EI\ \Delta_H/L^2 = \frac{6(500)\ EI}{EI\ 400} = 7.5$ ft. kips

and $M^F_{AB} = -7.5 = M^F_{BA}$

$$M^F_{CD} = +7.5 = M^F_{DC}$$

Solve for moments at the joints caused by the assumed spread.

JOINT	A		B					C			D
MEMB.	AB		BA	BC				CB	CD		DC
COF	+0.500				-0.343				+0.500		
K	0.20		0.20	0.3943	Balancing Moms.			0.3943	0.20		0.20
$\frac{K}{\Sigma K}$	—		0.337	0.663	At B		At C	0.663	0.337		—
FEM	-7.50		-7.5	+80.75				-80.75	+7.5		+7.5
	-12.38		-24.75	-48.50	-48.50			+16.65			
				-12.88			+37.50	+37.50	+19.10		+9.55
	+2.18		+4.36	+8.52	+8.52			-2.92			
				-0.66			+1.94	+1.94	+0.98		+0.49
	+0.11		+0.22	+0.44	+0.44			-0.15			
							+0.10	+0.10	+0.05		+0.02
M_{sp}	-17.59		-27.67	+27.67				-27.63	+27.63		+17.56
BAL. MOMS.					-39.54		+39.54				

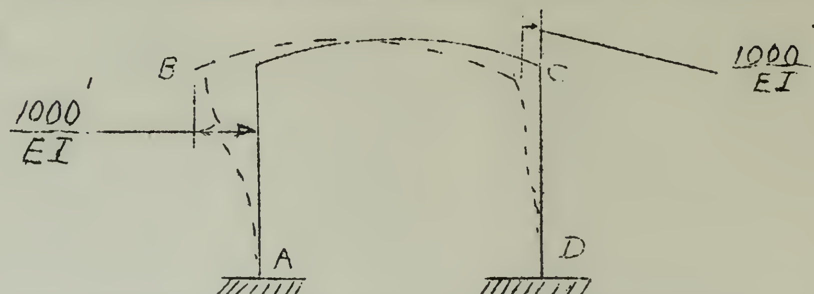
The consistent joint force (CJF) required at B and C to maintain the assumed spread is equal to the horizontal restraining forces at the ends of the arch.

HRF caused by:	HRF_L (+ →)	HRF_R
(a) Spread in the arch	-30.95 K	+30.95 K
(b) Balancing moment at joint B .21332(39.54)	+ 8.50	- 8.50
(c) Balancing moment at joint C	+ 8.50	- 8.50
(d) Shear in member AB $\frac{17.59 + 27.67}{20}$	- 2.26	
(e) Shear in member BC		+ 2.26
CJF_{sp}	-16.21 K ←	+16.21 K →

Thus a consistant joint force of 16.21 kips at each end of the arch acting away from each other will produce the given spread and since an artificial joint restraint of 7.02 kips was required to restrain the original structure from spreading, the actual spread will cause the moments at the various joints to change by an amount equal to

$$\frac{AJR_{sp}}{CJF_{sp}}(M_{sp}) = \frac{7.02}{16.21} M_{sp} = 0.432\ M_{sp}$$

Next we will assume a sidesway of 1000/EI feet to the left



The moments induced in the ends of the column members equals

$$\frac{6EI\Delta_H}{L^2} = \frac{6EI}{400}\ \frac{1000}{EI} = 15 \text{ foot kips}$$

$M^F_{AB} = -15$ foot kips

$M^F_{BA} = -15$ foot kips

$M^F_{CD} = -15$ foot kips

$M^F_{DC} = -15$ foot kips

Solve for moments at the joints caused by the assumed sidesway.

JOINT	A	B				C		D
MEM.	AB	BA	BC			CB	CD	DC
COF	+0.500		-0.343				+0.500	
K	0.20	0.20	0.3943	Balancing Moment		0.3943	0.200	0.200
$\frac{K}{\Sigma K}$	—	0.337	0.663	At B	At C	0.663	0.337	—
FEM	-15.00	-15.00					-15.00	-15.00
	+2.52	+5.05	+9.95	+9.95	+12.22	-3.42 +12.22	+6.20	+3.10
			-4.19			-0.96		
	+0.71	+1.41	+2.78	+2.78		+0.63	+0.33	+0.16
			-0.22					
	+0.03	+0.07	+0.15	+0.15				
M_{ss}	-11.74	-8.47	+8.47			+8.47	-8.47	-11.74
BAL. MOMS.				+12.88	+12.85			

The consistant joint forces, CJF, required at B and C to maintain the given sidesway are equal to the horizontal restraining forces at C and B and are equal to:

HRF caused by:	HRF_L (+ →)	HRF_R
(a) Balancing moment at joint B (0.21442)(12.88)	-2.76^K	$+2.76^K$
(b) Balancing moment at joint C	+2.76	-2.76
(c) Shear in member AB $\frac{(11.74)+(8.47)}{20}$	-1.01	
(d) Shear in member CD		-1.01
CJF_{ss}	-1.01^K ←	-1.01^K ←

Thus a consistant joint force of 1.01 kips at each end of the arch acting toward the left, will produce the given sidesway, and since an artificial joint restraint of 1.26 kips acting toward the right was required to keep the original structure from swaying, the actual sidesway will cause the moments at the various points to change by an amount equal to $\frac{AJR_{ss}}{CJF_{ss}} M_{ss} = \frac{1.26}{1.01} = 1.248\ M_{ss}$

TABULATION OF RESULTS Moments in foot kips.

FACTOR	M_{AB}	M_{BA}	M_{BC}	M_{CB}	M_{CD}	M_{DC}
M_R	+ 1.89	+ 3.77	- 3.77	- 3.13	+ 3.13	+ 1.56
0.432 M_{sp}	- 7.60	-11.92	+11.92	-11.92	+11.92	+ 7.60
1.248 M_{ss}	-14.65	-10.58	+10.58	+10.58	-10.58	-14.65
Final Mom.	-20.36	-18.73	+18.73	- 4.47	+ 4.47	- 5.49
Check by the Conjugate ~~Structure~~	-20.40	-18.40	+18.40	- 4.70	+ 4.70	-5.80

$$H_A = \frac{(20.36 + 18.73)}{20} = 1.95 \text{ kips} \longrightarrow$$

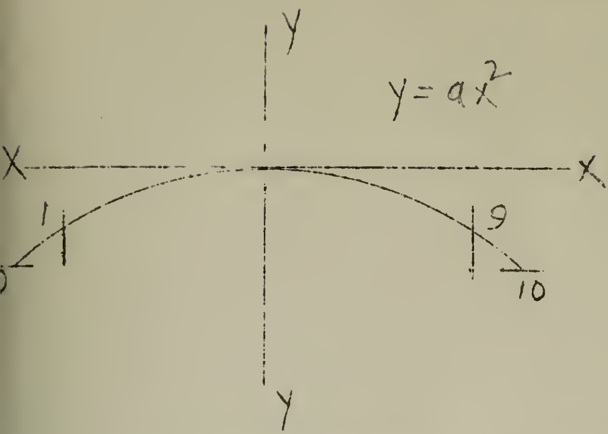
$$H_D = \frac{(5.49 - 4.47)}{20} = 0.05 \text{ kips} \longrightarrow$$

$$V_A = \frac{(40.00 + 60.00 - 20.40 - 5.80)}{20} = 3.69 \text{ kips} \uparrow$$

$$V_D = \frac{(140.00 - 40.00 + 20.40 + 5.80)}{20} = 6.31 \text{ kips} \uparrow$$

APPENDIX A

CALCULATIONS FOR ARC LENGTHS
METHOD A



Infinite series basic equation-

(1) arc lgth 0-10 = $L\left(1 + 2/3\left(\frac{2R}{L}\right)^2 - 2/5\left(\frac{2R}{L}\right)^4 + \text{etc...}\right.$

(2) arc lgth 1-9 = L(1 + 2/3 etc. etc. etc.

Subtract (2) from (1) and divide result by two to get length of segment 0-1

20' Span Rise/Span of 0.2

SEG.	L = 2X	R = y	$\frac{2R}{L}$	$+ 2/3(\frac{2R}{L})^2$	$-2/5(\frac{2R}{L})^4$	$+ 2/7(\frac{2R}{L})^6$	Factor
4-5	4	.16	.08	0.0042666	0.0000164	.0000048	1.00425
3-4	8	.64	.16	0.0170666	0.0002621	.000005	1.01681
2-3	12	1.44	.24	0.0383999	0.0013271	.000055	1.03713
1-2	16	2.56	.32	0.0682666	0.0041943	.0003067	1.0643790
0-1	20	4.00	.40	0.1066666	0.0102400	.0011702	1.09760

Total Arc L	$L_a - L_b$	$\frac{L_a - L_b}{2}$	Segment
21.951936			
17.0300640	4.921872	2.460936	0-1
12.4455288	4.584534	2.292267	1-2
8.1344744	4.311054	2.155527	2-3
4.0170008	4.017000	2.058737	3-4
	4.017000	2.00850	4-5

APPENDIX B

COMPUTATIONS FOR ARC LENGTHS
METHOD B

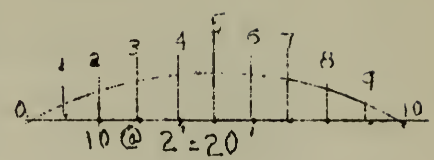
Basic Expression-Arc Length $= \frac{L^2}{8d} \ln\left(\frac{4d + \sqrt{16d^2 + L^2}}{L}\right) + \frac{\sqrt{16d^2 + L^2}}{2}$

20' Span 6.0' Rise Rise/Span-0.3

ARC SEG.	L	D	$16d^2$ +	L^2 =	$\sqrt{\ }$	$\sqrt{16d^2 + L^2}$	$\frac{4d + \sqrt{16d^2 + L^2}}{L}$	ln ( )	$\frac{L^2}{8d}$	$\frac{L^2}{8d}$ ln ( )	ln() + $\frac{\sqrt{\ }}{2}$
0-10	20	6.000	576	400	976	31.24099	2.762049	1.015974	8.333	8.46645	24.0869
1-9	16	3.840	235.929	256	491.92	22.17948	2.34622	0.85280	8.333	7.10666	18.1964
2-8	12	2.16	74.649	144	218.65	14.78680	1.95223	0.66896	8.333	5.57471	12.96811
3-7	8	0.96	14.746	64	78.75	8.87388	1.589235	0.46325	8.333	3.860417	8.2974
4-6	4	0.24	.922	16	16.92	4.11358	1.268395	0.23775	8.333	1.98125	4.0380

Total Arc L	$L_a - L_b$	$\frac{L_a - L_b}{2}$	Segment
24.08695			
18.19641	5.89054	2.94527	0-1
12.96811	5.22830	2.61420	1-2
8. 29736	4.67075	2.33538	2-3
4.03804	4.25932	2.12966	3-4
		2.01902	4-5

TABULATION OF ARC LENGTHS
ALL SPANS



Rise/Span-0.04

SEGMENT	20' SPAN	30' SPAN
0-1	2.02066	3.03099
1-2	2.01262	3.01894
2-3	2.00648	3.00972
3-4	2.00240	3.00358
4-5	2.00034	3.00051
0-10	20.0850	30.12750
Rise/Span-0.08		
0-1	2.08112	3.121687
1-2	2.050371	3.075556
2-3	2.026073	3.039109
3-4	2.009271	3.013846
4-5	2.001325	3.002048
0-10	20.336320	30.504495
Rise/Span-0.12		
0-1	2.178466	3.267699
1-2	2.110352	3.165528
2-3	2.057472	3.086208
3-4	2.2.021372	3.032058
4-5	2.003068	3.004602
0-10	20.741460	31.112190

Rise/span-0.16

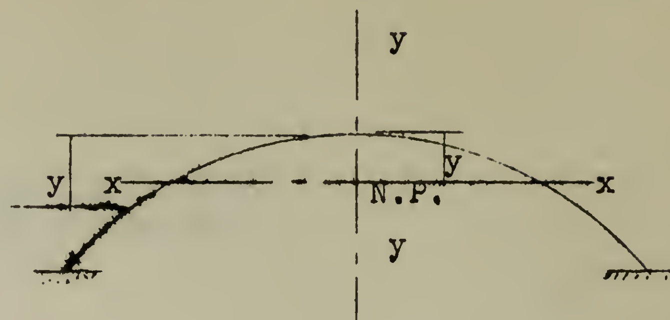
SEGMENT	20' SPAN	30' SPAN
0-1	2.304942	3.457414
1-2	2.215587	3.323380
2-3	2.076933	3.115400
3-4	2.037813	3.056720
4-5	2.005447	3.008171
10-0	21.281448	31.922172
Rise/Span-0.20		
0-1	2.460936	3.691404
1-2	2.292267	3.438402
2-3	2.155527	3.233292
3-4	2.058737	3.088104
4-5	2.008500	3.012750
0-10	21.951936	32.927904
Rise/Span-0.30		
0-1	2.94527	4.417905
1-2	2.6142	3.9213
2-3	2.33538	3.50307
3-4	2.12966	3.19449
4-5	2.01902	3.02853
0-10	24.08706	36.1306
Rise/Span-0.40		
0-1	3.507935	5.261902
1-2	3.005445	4.508168
2-3	2.565332	3.847998
3-4	2.224695	3.337042
4-5	2.033648	3.050472
0-10	26.674110	40.011166

APPENDIX C

COMPUTATIONS FOR NEUTRAL POINTS

20' Span Rise-0.8
Rise/Span-0.04

$y = ax^2$ $z = .008$



PT.	s = wI	x'	x'^{2}	y=ax^2	y x s	y'	Sx'^{2}	sy'	Sy'^{2}
1	2.0207	-9	81	.648	1.30941	-.3746	163.676	.7569	.2835
2	2.0127	-7	49	.392	.78897	-.1286	98.6223	.2588	.0333
3	2.0065	-5	25	.200	.40130	+.0634	50.1625	.1272	.0081
4	2.0024	-3	9	.072	.14417	+.1914	18.0216	.3833	.0734
5	2.0003	-1	1	.008	.01600	+.2554	2.0003	.5109	.1305
6									
7						I_{yy}	664.9668	I_{xx}	1.0576
8	Lower half of table is same as upper half.						based on Ad^2 values		
9	All values have been doubled in totals.								
10									

$$\frac{5.31972}{20.0850} = \bar{y} = 0.2649$$

TABULATION OF VALUES OF $\bar{Y}$
Centroid Distances

Rise / Span	Values of $\bar{y}$	
	20' Span	30' Span
0.04	0.26490	0.3970
0.08	0.53464	0.80150
0.12	0.81344	1.220146
0.16	1.10528	1.65540
0.20	1.40700	2.11050
0.30	2.22310	3.33465
0.40	3.10472	4.65708

CALCULATION OF TRUE MOMENTS OF INERTIA

C.G.I ≠ effect of rotation of axes

SECTION	LENGTH	$I_y = \frac{L^3}{12}$	x	$y = ax^2$	$y_a - y_b$	$\phi = \tan \frac{Q}{2}$	$\sin^2 \phi$ / $\cos^2 \phi$	$I_y \sin^2 \phi$	$I_y \cos^2 \phi$
0-1	2.0207	.6876	8	.512	.288	8°-11'-00"	.0202 / .9797	.0139	.6736
1-2	2.0127	.6793	6	.288	.224	6°-23'-00"	.0123 / .9876	.0084	.6709
2-3	2.0065	.6732	4	.128	.160	4°-34'-00"	.0063 / .9936	.0042	.6689
3-4	2.0024	.6692	2	.032	.096	2°-45'-00"	.0023 / .9976	.0015	.6676
4-5	2.0003	.6666	0	.000	.032	0°-55'-00"	.00023 / .9998	.00016	.6665
								0.0281	$6.7020 - I_{yy}$
								x 2	
								0.0562	

These corrections must be added to the Ad^2 values to get actual moments of inertia.

APPENDIX D

0	1	2	3	4	5	6	7	8	9	10	11
///////	ds	M/EI RECTANGLE				M/EI TRIANGLE				5 x 3 9 x 7	5 x 4 9 x 8
SECT.	LENGTH	HT.	y' dH	x' dV	θ 1 x 2	HT.	y' dH	x' dV	θ ½ x 1 x6	Δ_H RECT. AND TRIANGLE	Δ_V
0-1	3.03099										
1-2	3.01894					+3	-.1360	-10.0	+4.5284	- .6159	- 45.2840
2-3	3.00972	+ 3	+.0972	- 7.5	9.02916	+3	+.1358	- 7.0	+4.5146	+ .8776 + .6131	- 67.7187 - 31.6022
3-4	3.00358	+ 6	+.2891	- 4.5	18.02148	+3	+.3117	- 4.0	+4.5054	+ 5.2100 + 1.4044	- 81.0967 - 18.0216
4-5	3.00051	+ 9	+.3851	- 1.5	27.00459	+3	+.3917	- 1.0	+4.5008	+10.3995 + 1.7630	- 40.5069 - 4.5008
5-6	3.00051	+12	+.3851	+ 1.5	36.00612	+3	+.3757	+ 2.0	+4.5008	+13.8659 + 1.6909	+ 54.0092 + 9.0016
6-7	3.00358	+15	+.2891	+ 4.5	45.0537	+3	+.2637	+ 5.0	+4.5054	+13.0250 + 1.1881	+202.7417 + 22.5270
7-8	3.00972	+18	+.0972	+ 7.5	54.17496	+3	+.0559	+ 8.0	+4.5146	+ 5.2658 + .2524	+406.3122 + 36.1168
8-9	3.01894	+21	-.1906	+10.5	63.39774	+3	-.2479	+11.0	+4.5284	-12.0836 - 1.1227	+665.6763 + 49.8124
9-10	3.03099	+24	-.5744	+13.5	72.74376	+3	-.6477	+14.0	+4.5465	-41.7840 - 2.9447	+982.0408 + 63.6510
					325.43151		$\theta = +366.0764$		40.6449	Δ_H = -2.9952	+2203.1581 Δ_V

1 KIP VERTICAL AT POINT 1.

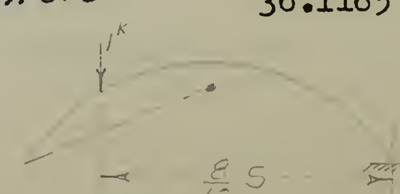
Rise-1.2'
Span-30'   Rise/Span-0.04   a= 0.00533
y= 0.3970

$\frac{9}{10}$ 5

0	1	2	3	4	5	6	7	8	9	10	11
/////	ds	M/EI RECTANGLE				M/EI TRIANGLE				5 x 3 9 x 7	5 x 4 9 x 8
SECT.	LENGTH	HT.	y′ dH	x′ dV	θ 1 x 2	HT.	y′ dH	x′ dV	θ ½ x 1x6	Δ_H RECT. AND	Δ_V TRIANGLE
0-1											
1-2											
2-3	3.00972					+ 3.0	+.1358	- 7.0	+4.5146	+ .6131	- 31.6022
3-4	3.00358	+ 3	+.2891	- 4.5	+ 9.0107	+ 3.0	+.3117	- 4.0	+4.5054	+ 2.6049 + 1.4044	- 40.5482 - 18.0216
4-5	3.00051	+ 6	+.3851	- 1.5	+18.0031	+ 3.0	+.3917	- 1.0	+4.5008	+ 6.9329 + 1.7630	- 27.0047 - 4.5008
5-6	3.00051	+ 9	+.3851	+ 1.5	+27.0046	+ 3.0	+.3757	+ 2.0	+4.5008	+10.3995 + 1.6906	+ 40.5069 + 9.0016
6-7	3.00358	+12	+.2891	+ 4.5	+36.0430	+ 3.0	+.2637	+ 5.0	+4.5054	+10.4200 + 1.1881	+162.1935 + 22.5270
7-8	3.00972	+15	+.0972	+ 7.5	+45.1458	+ 3.0	+.0559	+ 8.0	+4.5146	+ 4.3882 + .2524	+338.5935 + 36.1168
8-9	3.01894	+18	-.1906	+10.5	+54.3409	+ 3.0	-.2479	+11.0	+4.5284	-10.3574 - 1.1227	+570.5795 + 49.8124
9-10	3.03099	+21	-.5744	+13.5	+63.6508	+ 3.0	-.6477	+14.0	+4.5465	-36.5610 - 2.9447	+859.2858 + 63.6510
					253.1989	θ = 289.315			36.1165	-9.3284 Δ_H	+2030.5905 Δ_V

30' Span with Rise/Span of 0.04

1 KIP VERTICAL AT POINT 2



0	1	2	3	4	5	6	7	8	9	10	11
/////	ds	M/EI RECTANGLE				M/EI TRIANGLE				5 x 3 9 x 7	5 x 4 9 x 8
SECT.	LENGTH	HT.	y' dH	x' dV	θ 1 x 2	HT.	y' dH	x' dV	θ ½ x 1x6	Δ_H RECT. AND	Δ_V TRIANGLE
0-1											
1-2											
2-3											
3-4	3.00358					+ 3	+.3117	- 4	+4.5054	+ 1.4044	- 18.0216
4-5	3.00051	+ 3	+.3851	- 1.5	+ 9.0015	+ 3	+.3917	- 1	+4.5008	+3.4665 +1.7630	- 13.5023 - 4.5008
5-6	3.00051	+ 6	+.3851	+ 1.5	+18.0031	+ 3	+.3757	+ 2	+4.5008	+6.9329 +1.6909	+ 27.0047 + 9.0016
6-7	3.00358	+ 9	+.2891	+ 4.5	+27.0322	+ 3	+.2637	+ 5	+4.5054	+7.8150 +1.1881	+121.6449 + 22.5270
7-8	3.00972	+12	+.0972	+ 7.5	+36.1166	+ 3	+.0559	+ 8	+4.5146	+3.5105 + .2524	+270.8745 + 36.1168
8-9	3.01894	+15	-.1906	+10.5	+45.2841	+ 3	-.2479	+11	+4.5284	-8.6311 -1.1227	+475.4831 + 49.8124
9-10	3.03099	+18	-.5744	+13.5	+54.5578	+ 3	-.6477	+14	+4.5465	-31.3380 - 2.9447	+736.5303 + 63.6510
					189.9953				+31.6019	-16.0128 Δ_H	+1776.6216 Δ_V

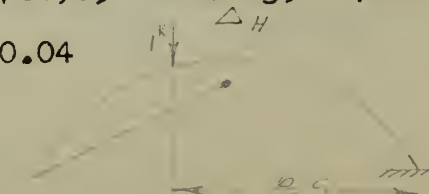
$\theta = 221.5972$

1 KIP VERTICAL AT POINT 3 30' Span rise/span-0.04

$\frac{7}{10}S$

0	1	2	3	4	5	6	7	8	9	10	11
//////	ds	M/EI RECTANGLE				M/EI TRIANGLE				5 x 3 9 x 7	5 x 4 9 x 8
Sect.	LENGTH	HT.	-y' dH	x' dV	θ 1 x 2	HT.	-y' dH	x' dV	θ ½ x 1x6	ΔH RECT. AND	ΔV TRIANGLE
0-1											
1-2											
2-3											
3-4											
4-5	3.00051					+ 3	+.3917	- 1.0	+4.5008	+1.7630	- 4.5008
5-6	3.00051	+ 3	+.3851	+ 1.5	+ 9.0015	+ 3	+.3757	+ 2.0	+4.5008	+ 3.4665 + 1.6909	+ 13.5023 + 9.0016
6-7	3.00358	+ [illegible]	+.2891	+ 4.5	+18.0215	+ 3	+.2637	+ 5.0	+4.5054	+ 5.2100 + 1.1881	+ 81.0968 + 22.5270
7-8	3.00972	+ 9	+.0972	+ 7.5	+27.0875	+ 3	+.0559	+ 8.0	+4.5146	+ 2.6329 + .2524	+203.1563 + 36.1168
8-9	3.01894	+12	-.1906	+10.5	+36.2273	+ 3	+.2479	+11.0	+4.5284	+ 6.9049 - 1.1227	+380.3867 + 49.8124
9-10	3.03099	+15	-.5744	+13.5	+45.4649	+ 3	-.6477	+14.0	+4.5465	-26.1150 - 2.9447	+613.7762 + 63.6510
					+135.8027	θ=162.9992			27.0965	-20.8835 ΔH	+1468.5263 ΔV

1 KIP VERTICAL AT POINT 4. 30' span Rise/Span = 0.04



0	1	2	3	4	5	6	7	8	9	10	11
/////	ds	M/EI RECTANGLE				M/EI TRIANGLE				5 x 3 9 x 7	5 x 4 9 x 8
Sect.	LENGTH	HT.	y′ dH	x′ dV	θ 1 x 2	HT.	y′ dH	x′ dV	θ ½ x 1x6	ΔH RECT. AND TRIANGLE	ΔV
5-6	3.00051					+ 3	+.3757	+ 2.0	+4.5008	+ 1.6909	+ 9.0016
6-7	3.00358	+ 3	+.2891	+ 4.5	+ 9.0107	+ 3	+.2637	+ 5.0	+4.5054	+ 2.6050 + 1.1881	+ 40.5482 + 22.5270
7-8	3.00972	+ 6	+.0972	+ 7.5	+18.0583	+ 3	+.0559	+ 8.0	+4.5146	+ 1.7553 + .2524	+135.4373 + 36.1168
8-9	3.01894	+ 9	-.1906	+10.5	+27.1705	+ 3	-.2479	+11.0	+4.5284	- 5.1787 - 1.1227	+285.2903 +491.8124
9-10	3.03099	+12	-.5744	+13.5	+36.3719	+ 3	-.6477	+14.0	+4.5465	-20.8920 - 2.9447	+491.0207 + 63.6510

+90.6114

$\theta = 5 + 9 = +113.2071$

- 0.9360

+22.5957 -21.7104 +1133.4053 = Δ_V

10 + 11 = -22.6464 = Δ_H

1 KIP Vertical at Point #5

1 KIP Vertical @ N.P. Rise- 1.2' a = 0.00533

y = 0.3970

0	1	2	3	4	5	6	7	8	9	10	11
//////	ds	M/EI RECTANGLE				M/EI TRIANGLE				5 x 3 / 9 x 7	5 x 4 / 9 x 8
Sect.	LENGTH	HT.	y' dH	x' dV	θ 1 x 2	HT.	y' dH	x' dV	θ $\frac{1}{2}$ x 1 x 6	Δ_H RECT. AND	Δ_V TRIANGLE
0-1	3.03099	-12	-.5744	-13.5	-36.3710	-3	-.6477	-14.0	-4.5465		+491.0207 + 63.6510
1-2	3.01894	- 9	-.1906	-10.5	-27.1705	-3	-.2479	-11.0	-4.5284		+285.2903 + 49.8124
2-3	3.00972	- 6	+.0972	- 7.5	-18.0583	-3	-.0559	- 8.0	-4.5146		+135.4373 + 36.1168
3-4	3.00358	- 3	+.2891	- 4.5	- 9.0107	-3	+.2637	- 5.0	-4.5054		+ 40.5482 + 22.5270
4-5	3.0005	/////	+.3851	- 1.5	//////	-3	+.3757	- 2.0	-4.5008		/////// + 9.0016
5-6	3.00051	/////	+.3851	+ 1.5	//////	+3	+.3757	+ 2.0	+4.5008		
6-7	3.00358	+ 3	+.2891	+ 4.5	+9.0107	+3	+.2637	+5 .0	+4.5054		
7-8	3.00972	+ 6	+.0972	+ 7.5	+18.0583	+3	+.0559	+ 8.0	+4.5146		
8-9	3.01894	+ 9	+.1906	+10.5	+27.1705	+3	-.2479	+11.0	+4.5284		
9-10	3.03099	+12	-.5744	+13.5	+36.3719	+3	-.6477	+14.0	+4.5465		
$\delta_{\theta M}$ =	30.12750				0.00				0.00		1133.4053 x2 2266.8106

δ_{VV} = 2266.8106

0	1	2	3	4	5	6	7	8	9	10	11
//////	ds	M/EI RECTANGLE------------				M/EI TRIANGLE------------				5 x 3 9 x 7	5 x 4 9 x 8
SECT.	LENGTH	HT.	y' dH	x' dV	θ 1 x 2	HT.	y' dH	x' dV	θ $\frac{1}{2}$ x 1x6	Δ_H RECT. AND TRIANGLE	
0-1	3.03099	+.3705	-.5744	-13.5	+1.1230	+.4324	-.64768	-14.0	+.6554	-.6451 -.4245	
1-2	3.01894	+.0347	-.1906	-10.5	+0.1048	+.3358	-.24793	-11.0	+.5069	-.0199 -.1257	
2-3	3.00972	.4356 2.5741				+.0347 -.2051	-.0209 +.1463	-8.854 -6.858	+.0075 -.2639	-.0020 -.0386	
3-4	3.00358	-.2051	+.2891	- 4.5	-.6161	-.1439	+.3117	-4.00	-.2161	-.1781 -.0674	
4-5	3.00051	-.3490	+.3851	- 1.5	-1.0473	-.0479	+.3917	-1.00	-.0719	-.4033 -.0282	
		ABOVE VALUES REPEAT IN LOWER HALF OF THIS TABLE.									
					+.8712			5 + 9	+1.2358 - .8712 =0.3646	-03.8620 δ_{HH}	0.0000

Rise = 1.2'
a = 0.00533
y = 0.3970

Span = 30'
Rise/Span = 0.04

1 KIP HORIZONTAL AT NEUTRAL POINT

	H_o			V_o		M_o		M_L	M_R
LOAD @ POINT	10	$H_o = \frac{dH}{dh}$	11	$V_o = -\frac{dV}{dv}$	0 = 5+9	7 $M_o = \frac{0}{\ }$	8= $(R-y)\ H_o$ $9=15V_o$	10 7+8-9	11 7+8+9+()
(27) 1	- 2.9952	- 0.7753	+2203.1	-0.9719	+366.076	-12.1509	- .6225 -14.5788	+1.8054	-0.3522
(24) 2	- 9.3284	-2.4145	+2030.5	-0.8958	+289.315	- 9.6030	- 1.9388 -13.4369	+1.8951	+.9787
(21) 3	-16.0128	-4.1446	+1776.6	-0.78375	+221.5972	- 7.3553	- 3.3281 -11.7563	+1.0729	-1.4397
(18) 4	-20.8835	-5.4053	+1468.5	-0.6478	+162.899	-5.4070	- 4.3405 - 9.7176	-0.0299	-1.4651
(15) 5	-22.6464	-5.8616	+1133.4	-0.5000	+113.207	-3.7576	- 4.7069 - 7.5000	-0.9645	-0.9645
6								-1.4651	-0.0299
7								-1.4397	+1.0729
8								-0.9787	+1.8951
9								-0.3522	+1.8054
Divisor	δ_{HH} 3.8620		δ_{VV} 2266.810		$\delta_{\theta M}$ 30.1275				

R= 1.2000
y= 0.3970
R-y =0.8030

30' Span
Rise/Span = 0.04

TABULATION OF CONSTANTS BASED ON A UNIT SPAN

Load Point	FEM (LEFT) Ft.Kips	FEM (RT.) ft.Kips	V (left)	H(thrust)
Rise/Span-0.04				
1	+.06036 S	-.01157 S	.9719	.7689
2	+.063325S	-.032465 S	.8958	2.4095
3	+.035915S	-.047835 S	.7838	4.1406
4	-.000855S	-.048695 S	.6478	5.4023
5	-.032025S	-.032025 S	.5000	5.8594
Rise/Span-0.08				
1	+.060000 S	-.011805 S	.9717	.3912
2	+.062680 S	-.032550 S	.8952	1.2119
3	+.035370 S	-.047715 S	.7831	2.0741
4	-.001055 S	-.048445 S	.6474	2.7002
5	-.031885 S	-.031885 S	.5000	2.9263
Rise/Span-0.20				
1	+.058855 S	-.011685 S	.9705	.1605
2	+.060300 S	-.031840 S	.8921	.4916
3	+.033425 S	-.045975 S	.7794	.8305
4	-.001300 S	-.046260 S	.6449	1.0720
5	-.030335 S	-.030335 S	.5000	1.1581
Rise/Span-0.30				
1	+.057900 S	-.011565 S	.9695	.1099
2	+.058525 S	-.030775 S	.8893	.3321
3	+.032060 S	-.043940 S	.7760	.5533
4	-.001162 S	-.043445 S	.6426	.7073
5	-.028210 S	-.028210 S	.5000	.7617
Rise/Span-0.40				
1	+.056945 S	-.011655 S	.9686	.0849
2	+.056165 S	-.030935 S	.8871	.2534
3	+.030500 S	-.043205 S	.7732	.4158
4	-.001630 S	-.042330 S	.6407	.5278
5	-.027615 S	-.027615 S	.5000	.5664

APPENDIX E

SAMPLE CALCULATIONS FOR DETERMINING:

(a) Absolute Stiffness

(b) Carry-over Factor

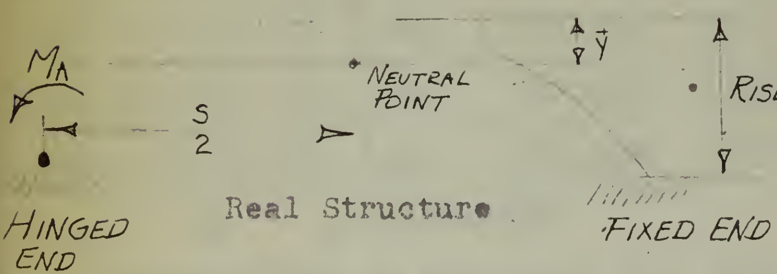
(c) Horizontal thrust due to rotation of the joint

(d) Effect of spread on the end reactions.

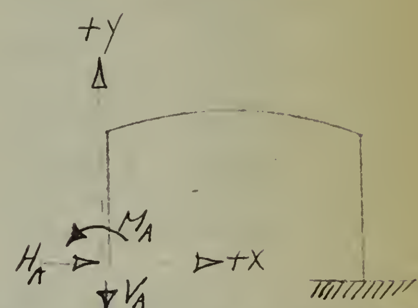
Data: 20' Span 0.08' Rise Rise/Span 0.04

Arc Lengths as previously calculated

Neutral point located 0.2649' below crown of arch.



Real Structure



Arch cut at the hinge and redundant forces H_A & V_A applied at joint.

From the theories of virtual work, the following relationships hold true:

$$EI\Delta_V = M_A(EI\delta_{VM}) + V_A\, EI\delta_{VV} + H_A(EI\delta_{VH}) \quad (4)$$

$$EI\Delta_H = M_A(EI\delta_{HM}) + V_A\, EI\delta_{HV} + H_A(EI\delta_{HH}) \quad (5)$$

$$EI\theta = M_A(EI\delta_{\theta M}) + V_A\, EI\delta_{\theta V} + H_A(EI\delta_{\theta H}) \quad (6)$$

FOR OUR SPAN

$$EI\delta_{VM} = EI\delta_{\theta V} = \Sigma \times S/2 = \frac{20.0850(10)}{} = 200.8500$$

$$EI\delta_{HM} = EI\delta_{\theta H} = \Sigma(RISE - \bar{y}) = (20.0850)(0.800 - 0.2649) = 10.74748$$

$$EI\delta_{VH} = EI\delta_{HV} = \Sigma(RISE - \bar{y})\left(\frac{S}{2}\right) = 20.0850(.5351)10 = 107.4748$$

$EI\,\delta_{VV} = \Sigma x^2 ds$ $EI\,\delta_{HH} = \Sigma y^2 ds$

ARC	ds in ft	x	x^2	x^2 ds	y	y^2	y^2 ds
0-1	2.0207	1	1	2.0207	.152	.0231	.0467
1-2	2.0127	3	9	18.1143	.408	.1665	.3350
2-3	2.0065	5	25	50.1625	.600	.3600	.7223
3-4	2.0024	7	49	98.1176	.728	.5300	1.0613
4-5	2.0003	9	81	162.0243	.792	.6273	1.2547
5-6	2.0003	11	121	242.0363	.792	.6273	1.2547
6-7	2.0024	13	169	338.4056	.728	.5300	1.0613
7-8	2.0065	15	225	451.4625	.600	.3600	.7223
8-9	2.0127	17	289	581.6703	.408	.1665	.3350
9-10	2.0207	19	361	729.4727	.152	.0231	.0467

EI = 2,673.4868 EI = 6.8400

Substituting in Equations (4), (5), & (6) we get,

$$EI\,\Delta_H = 200.8500\ M_A + 2{,}673.4868\ V_A + 107.4748\ H_A \quad (7)$$

$$EI\,\Delta_V = 10.7475\ M_A + 107.4748\ V_A + 6.8400\ H_A \quad (8)$$

$$EI\theta = 20.8500\ M_A + 200.8500\ V_A + 10.7475\ H_A \quad (9)$$

But since Δ_H & Δ_V are equal to zero, solving equations (7) and (8) simultaneously gives us: $V_A = -0.032474\ M_A$

$H_A = -1.06101\ M_A$

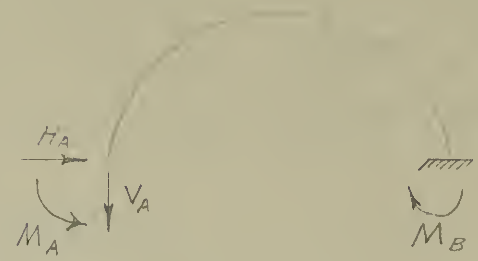
in which H_A is the horizontal thrust due to rotation of the joint and is equal to $-21.22\ M_A/S$.

Setting up the free body diagram shown below and solving for M_B

$M_B = M_A + V_A\, S$

$M_B = M_A - 0.64948\, M_A$

$M_B = 0.35052\, M_A$



By definition the carry-over factor is equal to the moment induced at the fixed end of a member when a unit moment is applied at the hinged end. Hence the

Carry-over factor $= M_B = -0.35052$

The carry over factor being negative since the induced moment at B causes tension on the same side of the member at the fixed end as the applied moment causes on the hinged end.

Substituting the values of M_A and H_A just obtained in equation (9), the following rotation at joint A occurs;

$EI\theta = 2.15942\, M_A$

and when θ is equal to one radian

$M_A = 0.46309\, EI$ (Absolute Stiffness)

$M_A = 9.2618\, \frac{EI}{S}$ (Absolute Stiffness)

EFFECT OF SPREAD

When spread occurs with no rotation or vertical deflection at the ends of the arch, equations (7) and (9) become equal to zero. Solving them simultaneously,

$$V_A = 0$$

$$H_A = -1.86881\ M_A$$

$$M_A = -0.53510\ H_A$$

Substituting in equation (8) and solving for H_A and M_A in terms of $EI\Delta_H$

$$M_A = -0.491328\ EI\Delta_H$$

$$H_A = +0.918206\ EI\Delta_H$$

Solving several span lengths, with the same rise/span ratio, it was determined that M_A varied inversely as the square of the span, and H_A varied inversely as the cube of the span. Therefore,

$$M_A = -196.53\ \frac{EI\Delta_H}{S^2}$$

$$H_A = +\ 7,345.6\ \frac{EI\Delta_H}{S^3}$$

rise/span	absolute stiffness	carryover
0.04	9.2618 EI/S	−0.3505
0.08	9.0522 EI/S	−0.3497
0.20	7.8855 EI/S	−0.3431
0.30	6.7742 EI/S	−0.3384
0.40	5.8205 EI/S	−0.3362

When joint A is free to rotate, and a moment of M_a ft. kips is applied at A, the following thrusts and shears are induced at A.

rise/span	thrust kips	vertical shear kips
0.04	21.220 EI/S M_a	0.6495EI/S M_a
0.08	10.629 EI/S M_a	0.6503EI/S M_a
0.20	4.288 EI/S M_a	0.6569EI/S M_a
0.30	2.894 EI/S M_a	0.6616EI/S M_a
0.40	2.203 EI/S M_a	0.6638EI/S M_a

When a spread of Δ_H feet occurs, the following moments, shears, and thrust occur:

rise/span	Thrust	Moment	Shear
0.04	7346	−196.5312	0.00
0.08	1806.3	−96.2225	0.00
0.20	247.64	−32.3053	0.00
0.30	103.83	−19.6084	0.00
0.40	52.391	−12.8234	0.00

NOTE: recorded values of thrust must be multiplied by $\frac{EI\,\Delta_H}{S^3}$

and all moments by $\frac{EI\,\Delta_H}{S^2}$

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H2 Hansen
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